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Transfer Matrix Spectrum and Bound States for Lattice Classical Ferromagnetic Spin Systems at High Temperature

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We obtain new properties of general d-dimensional lattice ferromagnetic spin systems with nearest neighbor interactions in the high-temperature region ($\beta \ll 1$). Each model is characterized by a single-site *a priori* spin distribution, taken to be even. We state our results in terms of the parameter $\alpha = \langle s^4 \rangle - 3\langle s^2 \rangle^2$, where $\langle s^k \rangle$ denotes the kth moment of the *a priori* distribution. Associated with the model is a lattice quantum field theory which is known to contain particles. We show that for $\alpha > 0$, β small, there exists a bound state with mass below the two-particle threshold. The existence of the bound state has implications for the decay of correlations, i.e., the 4-point functions decay at a slower rate than twice that of the 2-point function. These results are obtained using a lattice version of the Bethe–Salpeter equation. The existence results generalize to N-component models with rotationally invariant *a priori* spin distributions.

KEY WORDS: Transfer matrix spectrum; decay of correlations; bound states; Gaussian domination inequalities; classical ferromagnetic spin systems.

I. INTRODUCTION AND RESULTS

In this work we obtain new properties of general d-dimensional lattice ferromagnetic classical spin systems with nearest neighbor interactions in the high temperature region ($\beta \ll 1$). Each such system is characterized by a single site *a priori* spin probability distribution. Associated with these

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systems is a lattice quantum field theory with Hamiltonian energy and field momentum operators living on a $d-1$-dimensional sublattice. The Hamiltonian is minus the logarithm of the transfer matrix (see refs. 1 and 2). The new properties are uncovered by a detailed study of the particles of this underlying field theory. The idea of studying these systems via the transfer matrix is not new but up to now it has only been established that the low-lying energy-momentum (e-m) spectrum consists of a particle with isolated dispersion curve. These results imply exponential decay of correlation functions (cf) and the Ornstein–Zernike behavior of the two-point cf. refs. 3 and 4. Our results go beyond this giving information on the spectrum up to the two-particle threshold and have consequences for the decay of cf's.

Our basic result can be stated in terms of the quantity $\alpha \equiv \langle s^4 \rangle - 3\langle s^2 \rangle^2$ where the brackets are moments of the *a priori* distribution, taken to be even. We show that if $\alpha > 0$ the dominant interaction (which is local) is attractive and a bound state exists, such as, there is energy spectrum below the two-particle threshold. In the Gaussian case which corresponds to $\alpha = 0$ the particles do not interact. The presence of bound states in the spectrum imply decay properties of cf's, for example, the 4-point function has a slower than two particle decay rate.

The spectral results established here are obtained using a lattice version of the Bethe–Salpeter (B–S) equation (which employs a newly devised set of coordinates suitable for the lattice two-body bound state problem) and follow the methods used in refs. 5 and 6.

We point out that for a wide class of models cf inequalities have been established (see refs. 1, 7, and 8) called Gaussian domination inequalities. These inequalities hold for all temperatures. The class of models is defined by imposing conditions on the single spin distribution (ssd). For these models if we set the inverse temperature to zero and take coincident points then the 4-point inequality corresponds to the ssd condition $\alpha \leqslant 0$. Thus for high temperatures our results complement the Gaussian domination ones. In addition our bound state results generalize to N-component spin models (see below) while Gaussian domination inequalities have only been proven to hold for the scalar and Abelian ($N=2$) cases.

For the N-component spin models, where $s_i(x)$ is the ith component of $s(x) \in R^N$ and x is a lattice site of Z^d, the ssd is taken to be even and rotationally invariant. α, the parameter for the scalar spin case is replaced by $\alpha_N = \langle (s \cdot s)^2 \rangle - ((N+2)/N) \langle s \cdot s \rangle^2$ and a bound state exists for $\alpha_N > 0$. $\alpha_N = 0$ corresponds to the Gausian case. We now turn to a more precise description of the class of models we treat and of our results. For simplicity we only consider explicitly the scalar spin case. We let $s(x) \in R$, $x = (x_0, \vec{x}) \in \Lambda \subset Z^d$ denote the spin variable at the site x of the finite lattice Λ. For the generating function $Z_\Lambda(J)$ we take $Z_\Lambda(J) = \int e^{(J,s)} e^{S(s)}\, d\mu(s)$; $(J, s) = \sum_x J(x)\, s(x)$ and the interacting action $S(s)$ is $S(s) = \beta \sum' s(x)\, s(y)$ where $\sum'$ denotes the sum over the unordered set of nearest neighbor sites $\{x, y\}$. $d\mu(s) = \prod_x e^{-V(s(x))}\, ds(x)$ and we only consider the case of even ssd. i.e., $V(s) = V(-s)$. $V(s)$ is bounded from below and increases at infinity at least quadratically. Expectations of the probability measure $\exp[S(s)]\, d\mu(s)/$ normalization are denoted by $\langle \cdot \rangle_\Lambda$. Truncated cf's are given by local derivatives with respect to J's of $\ln Z_\Lambda(J)$ at $J = 0$.

By the polymer expansion (see ref. 4) the thermodynamic limit ($\Lambda \to Z^d$) of the cf's exist. The limiting cf's are denoted by $\langle \cdot \rangle$ and are translation invariant. The truncated cf's have exponential tree decay.

Associated with the model is an imaginary discrete time lattice quantum field theory (qft). The qft is constructed in the standard way (see refs. 1 and 2). Taking the x_0 direction as time the construction provides the quantum mechanical Hilbert space H with inner product $(\cdot, \cdot)$, commuting self-adjoint energy-momentum (em) operators $H \geqslant 0$, $\vec{P}$, the time-zero field operator $\hat{s}(x)$, $x = (0, \vec{x})$ and the vacuum vector Ω. The relation of the Hilbert space objects to the cf's is given by the Feynman–Kac (F–K) formula, i.e., setting $\hat{s}(0) = \hat{s}$, $x_k = (t_k, \vec{x}_k)$, with $t_1 \leqslant t_2 \leqslant \cdots \ll t_n$,

$$(\Omega, \hat{s} e^{-H(t_2 - t_1)} e^{i\vec{P} \cdot (\vec{x}_2 - \vec{x}_1)} \hat{s} e^{-H(t_3 - t_2)} e^{i\vec{P} \cdot (\vec{x}_3 - \vec{x}_2)} \hat{s} e^{-H(t_n - t_{n-1})} e^{i\vec{P} \cdot (\vec{x}_n - \vec{x}_{n-1})} \hat{s}\Omega)$$

$$= \langle s(x_1) \cdots s(x_n) \rangle$$

We will state our main result in terms of the spectrum of H, $\vec{P}$ but first we give some known or easily obtained results on the e-m spectrum which we need here. We let $(E, \vec{p})$, $E \geqslant 0$, $\vec{p} \in T_{d-1}$ (the $d-1$-dimensional torus) denote the spectral parameters associated with $(H, \vec{P})$ and refer to the spectral point $(E, \vec{p} = \vec{0})$ as the mass spectrum.

The one-particle states are generated by vectors of the form $\hat{s}(\vec{x})\, \Omega$ and by the methods of ref. 4 have mass $m \sim \ln \beta$ for β small and an isolated real analytic dispersion curve $w(\vec{p}) \geqslant w(\vec{0}) \equiv m$. The e-m dispersion curve is determined as the zero of $\tilde{\Gamma}(p_0 = iw(\vec{p}), \vec{p})$ where $\tilde{\Gamma}(p)$ is the Fourier transform of $\Gamma(x, y)$. Throughout this paper we define the Fourier transform without factors of 2π. $\Gamma(x, y)$ is minus the convolution inverse of the two-point function $\langle s(x)\, s(y) \rangle = S(x, y)$. To lowest order in β

$$w(\vec{p}) = -\ln \beta - \ln \langle s^2 \rangle - 2\beta \langle s^2 \rangle + \beta \langle s^2 \rangle\, 2 \sum_{i=1}^{d-1} (1 - \cos p_i) + 0(\beta^2)$$

Furthermore there is no spectrum up to $-(2 - \varepsilon) \ln \beta$, $\varepsilon(\beta) > 0$, and $\varepsilon(\beta) \downarrow 0$ as $\beta \downarrow 0$. This is known as the upper mass gap property and implies the Orstein–Zernike behavior for the two-point function (see ref. 3).

The general representation for $S(x) \equiv S(x, 0)$ can be obtained by adapting the work of refs. 2 and 4 to give, for $\beta > 0$,

$$S(x) = \int_0^\infty \int_{T_{d-1}} e^{-E|x_0|} e^{i\vec{p}\cdot\vec{x}} \, d\sigma_{\vec{p}}(E) \, d\vec{p}$$

where

$$d\sigma_{\vec{p}}(E) = Z(\vec{p}, \beta) \, \delta(E - w(\vec{p})) \, dE + d\hat{\sigma}_{\vec{p}}(E)$$

and $d\sigma_{\vec{p}}(E)$ as well as $d\hat{\sigma}_{\vec{p}}(E)$ are positive measures. Thus $\tilde{S}(p)$, the Fourier transform of $S(x)$, is given by

$$\tilde{S}(p) = (2\pi)^{d-1} \frac{\sinh w(\vec{p}, \beta) \, Z(\vec{p}, \beta)}{\cosh w(\vec{p}, \beta) - \cos p_0} + (2\pi)^{d-1} \int_0^\infty \frac{\sinh E}{\cosh E - \cos p_0} \, d\hat{\sigma}_{\vec{p}}(E)$$

$d\hat{\sigma}_{\vec{p}}(E)$ has support in $(\bar{m}, \infty)$ where $\bar{m} = -(3 - \varepsilon') \ln \beta$, $\varepsilon' \downarrow 0$ as $\beta' \downarrow 0$ is a lower bound for the onset of the three-particle spectrum. $Z(\vec{p}, \beta) = \partial \tilde{\Gamma}(p_0 = i\chi, \vec{p})/\partial\chi\,|_{\chi = w(\vec{p})}$ is positive for $\vec{p}, \beta$ real.

Using the methods of ref. 4 we have the bounds $|S(x)| \leqslant c_1 \, |\beta/c_2|^{|x_0| + |\vec{x}|}$, and for $|x_0| > 1$, $|\Gamma(x)| \leqslant c_1 \, |\beta/c_2|^{3|x_0| + |\vec{x}|}$. Taking into account the explicit short distance behavior of $S(x)$ and $\Gamma(x)$, namely $S(0) = \langle s^2 \rangle + 0(\beta^2)$, $\Gamma(0) = -\langle s^2 \rangle^{-1} + 0(\beta)$, $\Gamma(x = (1, \vec{0})) = \beta + 0(\beta^2)$ along with the above bounds shows that $Z(\vec{p}, \beta)$ is jointly analytic in $\vec{p}, \beta$ and that $Z(\vec{p}, \beta) = \langle s^2 \rangle/(2\pi)^{d-1} + 0(\beta)$.

To determine the mass spectrum (e-m spectrum at $\vec{p} = 0$) in the interval $(m, 2m)$ we consider the states in the subspace generated by $\hat{s}(\vec{x}) \, \hat{s}(\vec{y}) \, \Omega$. The truncated 4-point function related to this state (after subtracting out the vacuum contribution) is

$$D(x_1 x_2; x_3 x_4) = \langle s(x_1) \, s(x_2) \, s(x_3) \, s(x_4) \rangle - \langle s(x_1) \, s(x_2) \rangle \langle s(x_3) \, s(x_4) \rangle$$

where $x_i = (t_i, \vec{x}_i)$. By translation invariance D depends only on the difference variables. We now introduce the newly-devised relative coordinates (ξ, η, τ) which are the substitute for the center of mass and relative coordinates used in the continuum (see ref. 10). Let $\xi = x_2 - x_1$, $\eta = x_4 - x_3$, $\tau = x_3 - x_2$ and we denote by p, q, k the respective Fourier transform variables. Writing $\xi = (\xi_0, \vec{\xi})$, etc. it follows that if $\xi_0 = \eta_0 = 0$ $D(\xi, \eta, \tau) = (\theta(-\vec{\xi}), \; e^{-H|\tau_0|} e^{i\vec{P}\vec{\tau}} \theta(\vec{\eta}))$ where $\theta(\vec{\eta}) = \hat{s}(\vec{0}) \, \hat{s}(\vec{\eta}) \, \Omega - (\Omega, \vec{s}(\vec{0}) \, \hat{s}(\vec{\eta}) \, \Omega) \, \Omega$. A calculation shows, with $f\colon Z^d \to C$ a function of space position only and letting $\tilde{f}(\vec{p})$ and $\tilde{D}(p, q, k)$ denote the Fourier transform of f and D

$$\int d^d p d^d q \bar{\tilde{f}}(\vec{p})\, \tilde{f}(\vec{q})\, \tilde{D}(p, q, k)$$

$$= \int_0^\infty \int_{T_{d-1}} \frac{\sinh E}{\cosh E - \cos k_0} (2\pi)^{3d+2}\, \delta(\vec{q} - \vec{k})\, d(\theta(f), \mathsf{E}(E, \vec{q})\, \theta(f)) \tag{1.2}$$

where $\mathsf{E}(E, \vec{q})$ is the spectral family associated with H, $\vec{P}$ and T^{d-1} is the $d-1$-dimensional torus, $\theta(f) = \sum_{\vec{x}} f(\vec{x})\, \theta(-\vec{x})$, $\vec{x} \in Z^{d-1}$. The singularities in k_0, for $\vec{k}$ fixed, of the left side are points in the e-m spectrum by considering the right side.

We can now state our results. We assume from now on that $\langle s^2 \rangle > 0$, $\alpha = \langle s^4 \rangle - 3\langle s^2 \rangle^2 > 0$ and set $\gamma = (\langle s^4 \rangle - 3\langle s^2 \rangle^2)(\langle s^4 \rangle - \langle s^2 \rangle^2)^{-1}$ so that $0 < \gamma < 1$. We have the

Theorem. For $\beta > 0$ and sufficiently small the first point m_b in the mass spectrum above m is isolated and is given by $m_b = 2m - |\ln(1-\gamma)| + 0(\beta)$. For $\delta > 0$ sufficiently small m_b is the only point in the mass spectrum in $(m, m_b + \delta\gamma)$.

Remarks.

1. The result generalizes to N vector models replacing α by α_N.

2. We expect that the methods of refs. 5 and 6 apply which exclude mass spectrum in $(m_b, 2m)$.

We now sketch the method of proof (which follows closely that of refs. 5 and 6) and give the intuitive picture for the result.

A B–S equation is introduced which in operator form is

$$D = D_0 + DKD_0, \qquad K = D_0^{-1} - D^{-1}$$

where

$$D_0(x_1 x_2 x_3 x_4) = \langle s(x_1)\, s(x_3) \rangle \langle s(x_2)\, s(x_4) \rangle + \langle s(x_1)\, s(x_4) \rangle \langle s(x_2)\, s(x_3) \rangle$$

K is called the B–S kernel. In terms of kernels in the relative coordinates $(\vec{\xi}, \vec{\eta}, \tau)$ and denoting the Fourier transform in τ by ^ we can write the B–S equation as

$$\hat{D}(\vec{\xi}, \vec{\eta}, k) = D_0(\vec{\xi}, \vec{\eta}, k) + \int \hat{D}(\vec{\xi}, \vec{\xi}', k)\, \hat{K}'(\vec{\xi}', \vec{\eta}', k)\, \hat{D}_0(\vec{\eta}', \vec{\eta}, k)\, d\vec{\xi}'\, d\vec{\eta}' \tag{1.3}$$

where $\hat{K}'(\vec{\xi}, \vec{\eta}, k) = \hat{K}(-\vec{\xi}, -\vec{\eta}, k)$. As we are interested in the mass spectrum we set $k = k_0$, $\vec{k} = 0$ and write $\hat{D}(k^0)$, etc., considering the operators as matrix operators in the even subspace of $\ell_2(Z^{d-1})$.

$\hat{K}$ is decomposed as $\hat{K} = \hat{L} + \hat{M}$ where $\hat{L}$ is local and β-independent and $\hat{M}$ is $0(\beta)$. $\hat{L}$ is obtained by expanding $D_0^{-1} - D^{-1}$ in powers of β and retaining only the constant term. We call $\hat{L}$ the ladder approximation and it is given by, in relative coordinates,

$$\hat{L}(\vec{\xi}, \vec{\eta}, k) = \frac{\gamma}{2\langle s^2 \rangle^2} \delta(\vec{\xi})\, \delta(\vec{\eta}), \qquad \rho = \gamma/(2\langle s^2 \rangle^2)$$

The bound on $\hat{M}$ is a crucial input and follows from the bound

$$|M(\vec{\xi}, \vec{\eta}, \tau)| \leqslant c_1 \left| \frac{\beta}{c_2} \right|^{3\,|\tau_0| + 1/2\,|2\vec{\tau} + \vec{\xi} + \vec{\eta}| + 1/2\,|\vec{\xi}| + 1/2\,|\vec{\eta}|}$$

$$|M(\vec{0}, \vec{0}, 0)| \leqslant c_0\, |\beta|$$

obtained in ref. 11. Using the representation for $S(x)$ we obtain a representation for $\hat{D}_0$ given by

$$\hat{D}_0(\vec{\xi}, \vec{\eta}, k)$$
$$= 2(2\pi)^{d-1} \int_0^\infty \int_0^\infty \int_{T_{d-1}} \frac{\sinh(E + E') \cos \vec{p} \cdot \vec{\xi} \cos \vec{p} \cdot \vec{\eta}}{\cosh(E + E') - \cos k_0}\, d\sigma_{\vec{p}}(E)\, d\sigma_{\vec{p}}(E')$$

We use the spectral parameter χ or $\varepsilon = 2m - \chi$ where $k_0 = i\chi$, $0 < \chi < 2m$. Roughly speaking, in terms of the parameter $z = -\varepsilon$ Eq. (1.3) corresponds to a lattice Schroedinger operator resolvent equation

$$(H - z)^{-1} = (H_0 - z)^{-1} - (H_0 - z)^{-1}\, V(H - z)^{-1}$$

where H_0 is a lattice Laplacian and the potential V is a sum of an attractive δ potential and a small, non-local but exponentially decaying potential.

For $0 < \operatorname{Re} \varepsilon < 2m$, and ε bounded away from zero then for all sufficiently small β, $\hat{D}_0(\varepsilon)$ and $\hat{K}(\varepsilon)\, \hat{D}_0(\varepsilon)$ are analytic and bounded; $\hat{K}(\varepsilon)\, \hat{D}(\varepsilon)$ is also compact ($\hat{K}$ is Hilbert–Schmidt). The analytic Fredholm theorem (see ref. 12) applies and the left hand side of Eq. (1.2) can be written

$$(f, Df) = (f, \hat{D}_0 (1 - \hat{K}\hat{D}_0)^{-1} f) \tag{1.4}$$

except for a discrete set of ε's with $\varepsilon = 0$ as the only possible accumulation point. Thus the singularities of Eq. (1.4) can only occur for f satisfying the eigenvalue equation

$$\hat{K}(\varepsilon)\, \hat{D}_0(\varepsilon)\, f = f \tag{1.5}$$

It will be seen that for small β the ladder approximation $\hat{L}$ to $\hat{K}$ and the product of 1-particle contributions to $\hat{D}_0$ are dominant. For the rank one operator $\hat{L}\hat{D}_0$ there is a unique real ε, call it ε_L, for which $\hat{L}\hat{D}_0 f = f$ and ε_L is a $0(\beta)$ correction to the $\beta = 0$ value $\varepsilon_0 = -\ln(1-\gamma)$. Taking into account all contributions to $\hat{K}\hat{D}_0$ give an $0(\beta)$ correction and an isolated unique real ε, call it ε_b, which satisfies Eq. (1.5). Thus there is a bound state mass $m_b = 2m - \varepsilon_b$ given by $m_b = -\ln(1-\gamma) + 0(\beta)$.

We now describe the organization of this paper. In Section II we treat the problem in the ladder approximation. In Section III we treat the full problem and prove the theorem. The generalization to the N-component model is given in Section IV and concluding remarks are made in Section V. In an appendix we obtain the relative coordinate and Fourier transform form of the B–S equation.

II. LADDER APPROXIMATION

In this section we treat the ladder approximation to the eigenvalue equation in $e\ell_2(Z^{d-1})$, the even subspace of $\ell_2(Z^{d-1})$. Throughout β is taken to be small. Explicitly, letting $\tilde{f}(\vec{p})$ denote the Fourier transform of $f(\vec{\xi})$, etc., the Fourier transform of $\hat{L}\hat{D}f = f$ is

$$\rho \int_{T_{d-1}} H(\vec{p}, \varepsilon)\, \tilde{f}(\vec{p})\, d\vec{p} = \tilde{f}(\vec{p})$$

where

$$H(\vec{p}, \varepsilon) = 2(2\pi)^{d-1} \int_0^\infty \int_0^\infty \frac{\sinh(E+E')}{\cosh(E+E') - \cosh(2m-\varepsilon)}\, d\sigma_{\vec{p}}(E)\, d\sigma_{\vec{p}}(E') \tag{2.1}$$

Using the decomposition of $d\sigma_{\vec{p}}(\cdot)$ we write $H(\beta, \varepsilon) = H_1 + H_2 + H_3$ where $H_i(\beta, \varepsilon) = \int_{T_{d-1}} H_i(\vec{p}, \varepsilon)\, d\vec{p}$ with

$$H_1(\beta, \varepsilon) = 2(2\pi)^{d-1} \int_{T_{d-1}} \frac{\sinh 2w(\vec{p})\, Z(p, \beta)^2}{\cosh 2w(\vec{p}) - \cosh(2m-\varepsilon)}\, d\vec{p}$$

$$H_2(\beta, \varepsilon) = 2(2\pi)^{d-1}\, 2 \int_0^\infty \int_{T_{d-1}} \frac{\sinh(E+w)\, Z(\vec{p}, \beta)}{\cosh(E+w) - \cosh(2m-\varepsilon)}\, d\hat{\sigma}_{\vec{p}}(E)\, d\vec{p}$$

$$H_3(\beta, \varepsilon) = 2(2\pi)^{d-1} \int_0^\infty \int_0^\infty \int_{T_{d-1}} \frac{\sinh(E+E')}{\cosh(E+E') - \cosh(2m-\varepsilon)}\, d\hat{\sigma}_{\vec{p}}(E)\, d\sigma_{\vec{p}}(E')$$

and for $\tilde{f}(\vec{p})=$ constant we have the condition $\rho H(\beta, \varepsilon)=1$. In order to control H_2 and H_3 we need a bound on $d\hat{\sigma}_{\vec{p}}(E)$ which is given by

Lemma II.1.

$$\hat{\sigma}_{\vec{p}}(0, \infty)=\int_0^\infty d\hat{\sigma}_{\vec{p}}(E)=0(\beta)$$

Proof. Using the spectral representation for $S(x_0=0, \vec{x})$ and taking the spatial Fourier transform gives

$$S(x=0)+\sum_{|\vec{x}|\geqslant 1} e^{-i\vec{q}\cdot\vec{x}}S(0, \vec{x})$$

$$=(2\pi)^{d-1}\int_0^\infty d\sigma_{\vec{q}}(E)=(2\pi)^{d-1}\,Z(\vec{q}, \beta)+(2\pi)^{d-1}\int_0^\infty d\hat{\sigma}_{\vec{q}}(E)$$

As $S(x=0)=\langle s^2\rangle+0(\beta^2)$, $|S(0, \vec{x})|\leqslant c_1\,|\beta/c_2|^{|\vec{x}|}$ and $Z(\vec{q}, \beta)=\langle s^2\rangle/(2\pi)^{d-1}+0(\beta)$ the result follows.

In the lemma below we establish some important properties of the $H_i(\beta, \varepsilon)$'s. Recall that $\gamma=\alpha/(\langle s^4\rangle-\langle s^2\rangle)$, $\varepsilon_0=-\ln(1-\gamma)$ and $\rho=\gamma/(2\langle s\rangle^2)^2$.

Lemma II.2. There exist $\delta>0$, $\beta_0>0$ sufficiently small, such that H_1 admits an analytic extension to the region $|\beta|<\beta_0$, $|\varepsilon-\varepsilon_0|<\delta\gamma$ and in this region

$$|\rho H_1-1|<2\delta\gamma, \qquad \rho\,\frac{\partial H_1}{\partial\varepsilon}=-\frac{(1-\gamma)}{\gamma}\,[1+0(\delta)]+0(\beta)$$

Furthermore H_2 and H_3 admit analytic extensions in ε to $|\varepsilon-\varepsilon_0|<\delta\gamma$ for $0<\beta<\beta_0$. In this region $|\rho H_i|\leqslant c_i\beta$, $i=2, 3$; $|\rho(\partial H_2/\partial\varepsilon)|\leqslant\beta^2\gamma$, $|\rho(\partial H_3/\partial\varepsilon)|\leqslant\beta^4\gamma$.

Proof. We first consider H_1 which we write as

$$H_1=2(2\pi)^{d-1}\int_{T_{d-1}}\frac{(1-e^{-4w})\,Z(\vec{p}, \beta)^2\,d\vec{p}}{(1-e^{2(m-w)-\varepsilon}-e^{-2m-2w+\varepsilon}+e^{-4w})}$$

Using the fact that $w=-\ln\beta+r(\beta, \vec{p})$ with $r(\beta, \vec{p})$ analytic and

$$m-w=r(\beta, \vec{p}=0)-r(\beta, \vec{p})=0(\beta)$$

shows that the exponentials are analytic in β. Writing $\varepsilon = \varepsilon_0 + \Delta\varepsilon$ we have $1 - e^{-\varepsilon} = \gamma - (1-\gamma)(e^{-\Delta\varepsilon} - 1)$. The first two terms of D, the denominator, we write as

$$1 - e^{-\varepsilon} + e^{-\varepsilon} 2(w-m) \int_0^1 e^{-\sigma 2(w-m)}\, d\sigma$$

so that

$$\begin{aligned} D &= \gamma \left[1 + 0\left(\frac{1-\gamma}{\gamma}\right) |\Delta\varepsilon|\, e^{|\Delta\varepsilon|} + \frac{2\,|\beta|}{\gamma(1-\gamma)}\, e^{|\Delta\varepsilon|} + 0(2\beta^4) \right] \\ &\equiv \gamma(1 + r_d) \end{aligned}$$

Concerning $Z(\vec{p}, \beta)^2$ we write $Z(\vec{p}, \beta)^2 = Z_0^2 + \Delta N$ where $Z_0 = \langle s^2 \rangle / (2\pi)^{d-1}$ and $|\Delta N| \leqslant c_2 |\beta|$. Thus $\rho H_1 = 1 + (2(2\pi)^{d-1}/\gamma)\, \rho Z_0^2 \int (rd/(1+rd))\, d\vec{p} + (2(2\pi)^{d-1}/\gamma)\, \rho \int (\Delta N/(1+rd))\, d\vec{p}$. Taking β_0 as the largest $|\beta|$ satisfying $|\beta| < \delta\gamma^2/8$, $|\beta| < \gamma(1-\gamma)/4$, $c_2\, |\beta| < \delta\gamma \langle s^2 \rangle^2/4$ we have the bound $|\rho H_1 - 1| < 2\delta\gamma$. The integrands of H_2 and H_3 can be bounded in the same way and for $\beta > 0$, H_2 and H_3 are bounded using $\int_0^\infty \int_{T_{d-1}} d\hat{\sigma}_{\vec{p}}(E)\, d\vec{p} \leqslant c\, |\beta|$. Similarly the $\partial H_i / \partial\varepsilon$ are bounded.

We set $F = \rho H$, $F_i = \rho H_i$, $i = 1, 2, 3$. For $\alpha > 0$ and β sufficiently small we now show there is a unique solution of $F(\beta, \varepsilon) = 1$ with ε near ε_0 which we denote by $\varepsilon_L(\beta)$ or ε_L. Thus in this approximation there is a bound state with mass $m_L = 2m - \varepsilon_L$. We write, for $|\Delta\varepsilon| < \delta\gamma$, and noting that $F_1(0, \varepsilon) = \gamma/(1 - e^{-\varepsilon})$, $F_1(0, \varepsilon_0) = 1$,

$$\begin{aligned} F(\beta, \varepsilon_0 + \Delta\varepsilon) - 1 &= F_1(0, \varepsilon_0 + \Delta\varepsilon) - F_1(0, \varepsilon_0) + R(\beta, \varepsilon_0 + \Delta\varepsilon) \\ &\equiv \Delta F_1(0, \varepsilon_0) + R(\beta, \varepsilon_0 + \Delta\varepsilon) \end{aligned}$$

where

$$\begin{aligned} \Delta F_1(0, \varepsilon_0) &= -(1-\gamma)(1 - e^{-\Delta\varepsilon})(1 - (1-\gamma)\, e^{-\Delta\varepsilon})^{-1} \\ R(\beta, \varepsilon) &= F_1(\beta, \varepsilon) - F_1(0, \varepsilon) + F_2(\beta, \varepsilon) + F_3(\beta, \varepsilon) \end{aligned}$$

Noting that the denominator of $\Delta F_1(0, \varepsilon_0)$ is positive for $|\Delta\varepsilon| < \gamma$, $\Delta F_1(0, \varepsilon_0) < 0 (> 0)$ for $\Delta\varepsilon > 0 (< 0)$, $F_1(\beta, \varepsilon_0 + \Delta\varepsilon)$ is continuous at $\beta = 0$ and the bounds $|F_i(\beta, \varepsilon_0 + \Delta\varepsilon)| \leqslant c\, |\beta|$, $i = 2, 3$ we have, for sufficiently small β, $F(\beta, \varepsilon_0 + \Delta\varepsilon) - 1 < 0 (> 0)$ for $\Delta\varepsilon > 0 (< 0)$. As $F(\beta, \varepsilon) - 1$ is monotone strictly decreasing and continuous in ε (actually analytic) there exists a unique $\varepsilon = \varepsilon_L$ such that $F(\beta, \varepsilon_L) - 1 = 0$.

Let $\varepsilon = \varepsilon_0 + z$, $|z| < \delta\gamma$, and $f_\beta(z) \equiv F(\beta, \varepsilon_0 + z) - 1$. From the z analyticity of $f_\beta(z)$ we have a Cauchy integral representation for the solution z_L

of $f_\beta(z_L)=0$ given by $z_L=(2\pi i)^{-1}\int_C ((z\, df_\beta(z)/dz)/f_\beta(z))\, dz$ where C is a circle, centered at $z=0$ and of radius slightly smaller than $\delta\gamma$. An analysis of the integral gives the bound $z_L=0(\beta)$ so the bound state mass is given by, with $\varepsilon_L=\varepsilon_0+z_L$, $2m-\varepsilon_L=2m-|\ln(1-\gamma)|+0(\beta)$.

III. EXISTENCE OF A BOUND STATE

Here we prove the theorem of Section I. In $e\ell_2(Z^{d-1})$ we consider the family of operators $T_\beta(\mu,\varepsilon)\equiv\hat{L}\hat{D}_0+\mu\hat{M}'\hat{D}_0$, $\beta\hat{M}'=\hat{M}$, for complex μ, $|\mu|<\mu_0$, μ_0 sufficiently small, and for $\mu=\beta$, $T_\beta(\beta,\varepsilon)=\hat{K}\hat{D}_0$. $\mu\hat{M}'\hat{D}_0$ will be treated as a small perturbation of $\hat{L}\hat{D}_0$. We want to apply the analytic Fredholm theorem to $\hat{K}\hat{D}_0$. Now $\hat{L}$ and $\hat{M}$ are compact and analytic as $\hat{L}$ is finite range and $\hat{M}$ is Hilbert–Schmidt. Also $|\hat{L}|=\rho$ and $|\hat{M}|<c\beta$ so we need a bound on $|\hat{D}_0|$. The operator $\hat{D}_0$ on the Fourier transform space is the multiplication operator $H(\vec{p},\varepsilon)$ of Eq. (2.1) and with our convention $|\hat{D}_0|=(2\pi)^{d-1}\sup_{\vec{p}\varepsilon T_{d-1}}|H(\vec{p},\varepsilon)|$. First we give some bounds on the $H_i(\vec{p},\varepsilon)$'s. We have

Lemma III.1. For β sufficiently small

(a) and for real ε, $0<\varepsilon\leqslant 2m$, $i=1,2,3$ $H_i(\vec{p},\varepsilon)>0$, $(\partial H_i/\partial\varepsilon)(\vec{p},\varepsilon)<0$;

(b) and for $0<\operatorname{Re}\varepsilon\equiv\varepsilon_r\leqslant 2m$, $i=1,2,3$ $|H_i(\vec{p},\varepsilon)|\leqslant H_i(\vec{p},\varepsilon_r)\geqslant 0$;

(c) and for $\varepsilon_0-\delta\gamma<\operatorname{Re}\varepsilon\leqslant\varepsilon_0+\ln 2$

(i)

$$H_1(\vec{p},\varepsilon)=\frac{2}{(2\pi)^{d-1}}\frac{\langle s^2\rangle^2}{1-e^{-\varepsilon}}[1+0(\delta)]+0(\beta)$$

$$\frac{\partial H_1}{\partial\varepsilon}(\vec{p},\varepsilon)=\frac{-2\langle s^2\rangle^2}{(2\pi)^{d-1}}\frac{e^{-\varepsilon}}{(1-e^{-\varepsilon})^2}[1+0(\delta)]+0(\beta)$$

(ii)

$$H_2(\vec{p},\varepsilon)=2\langle s^2\rangle\, 2\int d\hat{\sigma}_{\vec{p}}(E)(1+0(\beta))=0(\beta)$$

$$\frac{\partial H_2}{\partial\varepsilon}(\beta,\varepsilon)=2\langle s^2\rangle\, 0(\beta^2)\, e^{-\varepsilon}\int d\hat{\sigma}_{\vec{p}}(E)$$

(iii)

$$H_3(\vec{p}, \varepsilon) = 2(2\pi)^{d-1} \left(\int d\hat{\sigma}_{\vec{p}}(E) \right)^2 (1 + 0(\beta)) = 0(\beta^2)$$

$$\frac{\partial H_3}{\partial \varepsilon}(\vec{p}, \varepsilon) = 2(2\pi)^{d-1}\, 0(\beta^4)\, e^{-\varepsilon} \left(\int d\hat{\sigma}_{\vec{p}}(E) \right)^2$$

Proof. (a) follows by calculation. (b) write for χ, ψ real $\cosh(\chi + i\psi)$. Then, with $u \equiv E + E'$ and D denoting the denominator,

$$|D| = |\cosh u - \cosh(\chi + i\psi)| \geqslant \cosh u - \max_{\psi} |\cosh(\chi + i\psi)|$$

But $|\cosh(\chi + i\psi)|^2 = \cosh^2 \chi - \sin^2 \psi$ so $|D| \geqslant \cosh u - \cosh \chi = D$. (c) follows as in the proof of Lemma II.

We have

Lemma III.2. For $\varepsilon = \varepsilon_0 + \Delta\varepsilon$, $\Delta\varepsilon = \ln 2$ and β sufficiently small $|\hat{K}(\varepsilon)\, \hat{D}_0(\varepsilon)| < 1$.

Proof. We have $|\hat{K}\hat{D}_0| = |\hat{L}\hat{D}_0 + \hat{M}\hat{D}_0| \leqslant |\hat{L}|\, |\hat{D}_0| + |\hat{M}|\, |\hat{D}_0|$, with $|\hat{L}| = \rho$ and $|\hat{M}| \leqslant c\beta$. For $|\hat{D}_0|$ the contributions of H_2 and H_3 are of order β from Lemma II.1c. For H_1 we have $(2\pi)^{d-1} \sup_{\vec{p}} |H_1(\vec{p}, \varepsilon_0 + \Delta\varepsilon)| \leqslant 2\langle s^2 \rangle (1 - e^{-\varepsilon_0 - \Delta\varepsilon})^{-1} + 0(\beta)$. But

$$(1 - e^{-\varepsilon_0 - \Delta\varepsilon})^{-1} = (\gamma + (1 - \gamma)(1 - e^{-\Delta\varepsilon}))^{-1} = (\gamma + (1 - \gamma)/2)^{-1}$$

so that $|\hat{D}_0| = 2\langle s^2 \rangle (\gamma + (1 - \gamma)/2)^{-1} + 0(\beta)$ and

$$|\hat{L}\hat{D}_0| \leqslant |\hat{L}|\, |\hat{D}_0| \leqslant \gamma(\gamma + (1 - \gamma)/2)^{-1} + 0(\beta) < 1$$

Corollary. For $2m \geqslant \operatorname{Re} \varepsilon > \varepsilon_0 + \Delta\varepsilon$, $\Delta\varepsilon = \ln 2$ and β sufficiently small $|\hat{K}(\varepsilon)\, \hat{D}_0(\varepsilon)| < 1$.

Proof. Follows from Lemma III.2 and Lemma III.1 a) and b).

From Lemma III.1c, Lemma III.2 and the corollary $(1 - \hat{K}_0 \hat{D}_0)^{-1}$ exists in $\varepsilon_0 - \delta\gamma < \operatorname{Re} \varepsilon < \varepsilon_0 + \ln 2$.

We now return to the analysis of the operator $T_\beta(\mu, \varepsilon)$ which we treat as a perturbation of $T_\beta(0, \varepsilon_L)$ writing $T_\beta(\mu, \varepsilon) = T_\beta(0, \varepsilon_L) + \delta T_\beta(\mu, \varepsilon)$ where $\delta T_\beta(\mu, \varepsilon) = T_\beta(\mu, \varepsilon) - T_\beta(0, \varepsilon) + T_\beta(0, \varepsilon) - T_\beta(0, \varepsilon_L)$. Concerning $T_\beta(0, \varepsilon_L)$ we have (denoting the spectrum of an operator B by $\sigma(B)$)

Lemma III.3. For β sufficiently small $\sigma(T_\beta(0, \varepsilon_L)) \subset \{0, 1\}$.

Proof. Decomposing $\tilde{f}(\vec{p}) = \tilde{f}_0 + \tilde{f}_1(\vec{p})$ where $\tilde{f}_0$ is the constant function and $\tilde{f}_1(\vec{p})$ is orthogonal to the constants we can write, letting $A_0 \equiv T_\beta(0, \varepsilon_L)$,

$$(\tilde{g}, T_\beta(0, \varepsilon_L)\, \tilde{f}) = (\tilde{g}_0, \tilde{g}_1) \begin{pmatrix} 1 & A_{01} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{f}_0 \\ \tilde{f}_1 \end{pmatrix}$$

where $(\tilde{g}_0, A_{01} \tilde{f}_1) = \bar{\tilde{g}}_0 \rho \int_{T_{d-1}} H(\vec{p}, \varepsilon_L)\, \tilde{f}_1(\vec{p})\, d\vec{p}$. For $\beta = 0$ the product of the one-particle contributions is zero. Using the maximum modulus theorem in β for $|\beta| < \beta_0$ and Lemma III.1c for the other contributions gives the bound $|A_{01}| < c'\beta$. The inverse $(A_0 - w)^{-1}$ is given by $(A_0 - w)^{-1} = \begin{pmatrix} (1-w)^{-1} & w^{-1}(1-w)^{-1} A_{01} \\ 0 & -w^{-1} \end{pmatrix}$ from which we see the spectrum is $\{0, 1\}$.

For the spectrum of $T_\beta(\mu, \varepsilon)$ we have

Lemma III.4. For β, μ_0, δ sufficiently small and for all μ, ε such that $|\mu| < \mu_0$, $|\varepsilon - \varepsilon_0| < 2\delta\gamma$,

$$\sigma(T_\beta(\mu, \varepsilon)) \subset \{w : |w-1| < \tfrac{1}{4} \text{ or } |w| < \tfrac{1}{4}\}$$

Proof. We have, from the proof of Lemma III.3, for $|w| > \frac{1}{4}$ and $|w - 1| > \frac{1}{4}$,

$$|(T_\beta(0, \varepsilon_L) - w)^{-1}| \leqslant \max\{|w|^{-1}, |1-w|^{-1}, |w|^{-1}\, |1-w|^{-1}\, |A_{01}|\} \leqslant 16$$

Using the Taylor expansion in ε and Lemma III.1c we have

$$|\hat{L}D_0(\beta, \varepsilon) - \hat{L}\hat{D}_0(\beta, \varepsilon_L)| \leqslant \frac{(1-\gamma)}{\gamma}\, |\varepsilon - \varepsilon_L| + 2\gamma\beta^2\, |\varepsilon - \varepsilon_L| + 0(\beta)\, |\varepsilon - \varepsilon_L|$$

and since $|\hat{D}_0| < (\langle s^2 \rangle^2/\gamma)(1 + 2\delta) + 0(\beta)$ and $|M'| < c'$, $|\mu \hat{M}' \hat{D}_0(\beta, \varepsilon)| \leqslant c2\, |\mu|$. Thus $|\delta T_\beta(\mu, \varepsilon)| \leqslant ((1-\gamma)/\gamma)\, |\varepsilon - \varepsilon_L| + 2\gamma\beta^2\, |\varepsilon - \varepsilon_L| + 2\, |\mu|\, c$ so that for β sufficiently small the Neumann series converges for $(T_\beta(\mu, \varepsilon) - w)^{-1}$.

Thus we have established the

Corollary. For β, δ sufficiently small and for all μ, ε such that $|\mu| < \beta$, $|\varepsilon - \varepsilon_L| < 2\delta\gamma$ the spectrum of $T_\beta(\mu, \varepsilon)$ consists of

(a) a simple eigenvalue $\alpha_\beta(\mu, \varepsilon)$, $|\alpha_\beta(\mu, \varepsilon) - 1| < \frac{1}{4}$ analytic in μ, ε and real for μ, ε real.

(b) Other spectrum in $|w| < \frac{1}{4}$.

Proof. (a) follows from analytic perturbation theory (see ref. 13) and also the multiplicity is one. For μ, ε real $T_\beta(\mu, \varepsilon)$ commutes with complex conjugation thus both $\alpha_\beta(\mu, \varepsilon)$ and $\bar{\alpha}_\beta(\mu, \varepsilon)$ are eigenvalues. As the multiplicity is one $\alpha_\beta(\mu, \varepsilon)$ is real.

Exploiting the μ, ε analyticity we now use Cauchy estimates to control derivatives and establish the existence of ε_b such that $\alpha_\beta(\beta, \varepsilon_b) = 1$. We have the

Theorem III.1. For β, δ sufficiently small and for all μ, $|\mu| < 2\beta$ there is a unique $\varepsilon_1(\mu)$, $|\varepsilon_1(\mu) - \varepsilon_0| \leqslant \delta^3\gamma$, such that $\alpha_\beta(\mu, \varepsilon_1(\mu)) = 1$.

Proof. For μ and ε Cauchy estimates we use the circles of radii $|\mu| = \mu_0$ and $|\varepsilon - \varepsilon_0| = 2\delta\gamma$, and take the μ, ε variables in the region $|\mu| < 2\beta$, $|\varepsilon - \varepsilon_0| < \delta^3\gamma$. By Cauchy estimates $|\partial_\mu \alpha_\beta(\mu, \varepsilon)| < 2/\mu_0$, $|\partial_\mu \partial_\varepsilon \alpha_\beta(\mu, \varepsilon)| < (2/\mu_0)(1/\delta\gamma)$ so that by Taylor expanding in μ and by Cauchy estimates $|\alpha_\beta(\mu, \varepsilon) - \alpha_\beta(0, \varepsilon)| \leqslant 4\beta/\mu_0$. Taking $\varepsilon = \varepsilon_L(\beta)$ we have $|\alpha_\beta(\mu, \varepsilon_L) - 1| \leqslant 4\beta/\mu_0$. For real ε

$$\partial_\varepsilon \alpha_\beta(\mu, \varepsilon) = \partial_\varepsilon \alpha_\beta(0, \varepsilon) + \int_0^\mu \partial'_\mu \, \partial_\varepsilon \alpha_\beta(\mu', \varepsilon) \, d\mu' \tag{3.1}$$

where

$$\partial_\varepsilon \alpha_\beta(0, \varepsilon) \leqslant -\frac{(1-\gamma)}{\gamma} + 0(2\beta^2\gamma) \tag{3.2}$$

and the 2nd term of Eq. (3.1) is bounded by $8\beta/\delta\gamma\mu_0^2$. Using Eq. (3.2) we have, for sufficiently small β, $\partial_\varepsilon \alpha_\beta(\mu, \varepsilon) \leqslant -((1-\gamma)/2\gamma)$. Thus

$$\begin{aligned} \alpha_\beta(\mu, \varepsilon_0 + \delta^3\gamma) &= \alpha_\beta(\mu, \varepsilon_0) + \partial_\varepsilon \alpha_\beta(\mu, \varepsilon_0) \, \gamma\delta^3 + \int_0^\varepsilon d\varepsilon' \int_0^{\varepsilon'} d\varepsilon'' \, \frac{\partial^2}{\partial \varepsilon''^2} \alpha_\beta(\mu, \varepsilon'') \\ &\leqslant 1 + 0(\beta) - \frac{(1-\gamma)}{2} \delta^3 + 0(\delta^4) \end{aligned}$$

and $\alpha_\beta(\mu, \varepsilon_0 + \delta^3\gamma) < 1$ for β sufficiently small. Similarly $\alpha_\beta(\mu, \varepsilon_0 - \delta^3\gamma) > 1$.

Taking $\mu = \beta$ and $\varepsilon_1(\beta) \equiv \varepsilon_b$ in Theorem III.1 gives us a bound state mass of $2m - \varepsilon_b = 2m - |\ln(1-\gamma)| + 0(\beta)$ and completes the proof of the theorem.

IV. *N*-COMPONENT VECTOR MODEL

We now consider N-component vector models with even, rotationally invariant ($O(N)$) ssd and interaction action. The spin variable at site $x \in Z^d$

is denoted by $s(x) \in R^N$ with components $s_i(x) \in R$, $i = 1, 2,..., N$ and the time zero operators are denoted by $\hat{s}(\vec{x})$, $\vec{x} \in Z^{d-1}$. The one-particle states are generated by vectors of the form $\hat{s}_k(\vec{x})\,\Omega$. The two-particle states are generated by $\hat{s}_k(\vec{x})\,\hat{s}_\ell(\vec{y})\,\Omega$ and these states can be decomposed into the rotationally invariant state $\hat{s}(\vec{x})\cdot\hat{s}(\vec{y})\,\Omega$ and the traceless states $(\hat{s}_k(\vec{x})\,\hat{s}_\ell(\vec{y}) - (1/N)\,\hat{s}(\vec{x})\cdot\hat{s}(\vec{y})\,\delta_{k\ell})\,\Omega$. The traceless states can further be decomposed into the symmetric and anti-symmetric states $(\hat{s}_k(\vec{x})\,\hat{s}_\ell(\vec{y}) + \hat{s}_\ell(\vec{x})\,\hat{s}_k(\vec{y}) - (2/N)\,\hat{s}(\vec{x})\cdot\hat{s}(\vec{y}))\,\Omega$ and $(\hat{s}_k(\vec{x})\,\hat{s}_\ell(\vec{y}) - \hat{s}_\ell(\vec{x})\,\hat{s}_k(\vec{y}))\,\Omega$, respectively. We only consider the rotationally invariant state. Associated with this state (after subtracting out the vacuum contribution) is the cf

$$D(x_1x_2x_3x_4) = \langle s(x_1)\cdot s(x_2)\,s(x_3)\cdot s(x_4)\rangle - \langle s(x_1)\cdot s(x_2)\rangle\langle s(x_3)\cdot s(x_4)\rangle$$

and the F–K formula and spectral representation equation remain valid. In the B–S eq. $D = D_0 + DKD_0$ we take

$$\begin{aligned} &ND_0(x_1x_2x_3x_4)\\ &\quad= \langle s(x_1)\cdot s(x_3)\rangle\langle s(x_2)\cdot s(x_4)\rangle + \langle s(x_1)\cdot s(x_4)\rangle\langle s(x_2)\cdot s(x_3)\rangle \end{aligned}$$

As before D and D_0 are decomposed into the diagonal and non-diagonal part, the non-diagonal parts are of order β. We find

$$\begin{aligned} D_d(x_1x_2x_3x_4) = {} & N[\langle s_1^4\rangle - N\langle s_1^2\rangle + (N-1)\langle s_1^2 s_2^2\rangle]\\ & \times \delta(x_3 - x_1)\,\delta(x_4 - x_2)\,\delta(x_2 - x_1)\\ & + N\langle s_1^2\rangle^2\,\delta(x_3 - x_1)\,\delta(x_4 - x_2)(1 - \delta(x_2 - x_1)) + 0(\beta)\\ D_{0d}(x_1x_2x_3x_4) = {} & 2N\langle s_1^2\rangle^2\,\delta(x_3 - x_1)\,\delta(x_4 - x_2)\,\delta(x_2 - x_1)\\ & + N\langle s_1^2\rangle^2\,\delta(x_3 - x_1)\,\delta(x_4 - x_2)(1 - \delta(x_2 - x_1)) + 0(\beta) \end{aligned}$$

and for $K = D_0^{-1} - D^{-1}$

$$\begin{aligned} K &= D_{0d}^{-1} - D_d^{-1} + 0(\beta)\\ &= \frac{1}{N}\left[\frac{\langle s_1^4\rangle - N\langle s_1^2\rangle + (N-1\langle s_1^2 s_2^2\rangle - 2\langle s_1^2\rangle^2}{2\langle s_1^2\rangle^2\,(\langle s_1^4\rangle - N\langle s_1^2\rangle^2 + (N-1)\langle s_1^2 s_2^2\rangle^2)}\right]\\ &\quad\times \delta(x_3 - x_1)\,\delta(x_4 - x_2)\,\delta(x_2 - x_1) + (0\beta) \end{aligned}$$

Dropping the $0(\beta)$ terms, rewriting in terms of s and taking the Fourier transform gives $\tilde{L}(\vec{p}, \vec{q}, k) = \alpha_N$ where α_N is given in the introduction. The rest of the analysis goes through as in the previous section.

V. CONCLUDING REMARKS

We have found a simple criteria for the existence of bound states based on the sign of α for small β. The question arises as to the existence and number of bound states for large values of β. Also there is the question of whether or not the result generalizes to the case of non-even ssd. For example, if α is calculated using zero average fields does the sign of α still determine the presence or absence of bound states. The existence of weakly bound, bound states in lattice gauge and gauge-matter models (strongly bound, bound states are present) is also an open question and the methods developed here open the way to treat these problems.

APPENDIX. LATTICE B–S EQUATION

Here we deduce a composition of kernel form for the Fourier transform of the lattice B–S eq. We use the relative and conjugate variables

$$\xi = x_2 - x_1,\, p; \qquad \eta = x_4 - x_3,\, q; \qquad \tau = x_3 - x_2,\, k$$

with $\xi_0 = 0$, $\eta_0 = 0$ and an integral notation for lattice sums. The B–S eq. is, with $x_{10} = x_{20}$, $x_{30} = x_{40}$

$$\begin{aligned} D(x_1 x_2 x_3 x_4) &= D_0(x_1 x_2 x_3 x_4) + \int dy_1\, dy_2\, dy_3\, dy_4\, \delta(y_{10} - y_{20})\, \delta(y_{30} - y_{40}) \\ &\quad \times D(x_1 x_2 y_1 y_2) K(y_1 y_2 y_3 y_4)\, D_0(y_3 y_4 x_3 x_4) \end{aligned} \tag{A.1}$$

All kernels are assumed to be translationally invariant. In terms of the relative variables ξ, η, τ we write, using a bar notation for the function of the relative variables,

$$\bar{D}(\xi, \eta, \tau) = D(0, x_2 - x_1 = \xi, x_3 - x_1 = \xi + \tau, x_4 - x_1 = \xi + \eta + \tau)$$

etc. The kernels D, D_0 and consequently also K are invariant under the substitutions

$$(x_1 x_2 x_3 x_4) \to (x_2 x_1 x_3 x_4) \tag{A.2a}$$

$$(x_1 x_2 x_3 x_4) \to (x_1 x_2 x_4 x_3) \tag{A.2b}$$

which imply

$$\bar{K}(\xi, \eta, \tau) = \bar{K}(-\xi, \eta, \tau + \xi) \tag{A.3a}$$

$$\bar{K}(\xi, \eta, \tau) = \bar{K}(\xi, -\eta, \tau + \eta) \tag{A.3b}$$

We introduce the variables $\xi', \eta', \tau', \tau''$ where

$$\xi' = y_2 - y_1 \tag{A.4a}$$

$$\eta' = y_4 - y_3 \tag{A.4b}$$

$$\tau' = y_1 - x_2 \tag{A.4c}$$

$$\tau'' = x_3 - y_4 \tag{A.4d}$$

Then

$$y_1 = \tau' + x_2 \tag{A.5a}$$

$$y_2 = \xi' + y_1 = \xi' + \tau' + x_2 \tag{A.5b}$$

$$y_4 = x_3 - \tau'' \tag{A.5c}$$

$$y_3 = y_4 - \eta' = x_3 - \tau'' - \eta' \tag{A.5d}$$

We have

$$D(x_1 x_2 y_1 y_2) = \bar{D}(\xi, \xi', \tau') \tag{A.6a}$$

$$D_0(y_3 y_4 x_3 x_4) = \bar{D}_0(\eta', \eta, \tau'') \tag{A.6b}$$

$$\begin{aligned} K(y_1 y_2 y_3 y_4) &= \bar{K}(\xi', \eta', \tau - \tau' - \tau'' - \xi' - \eta') \\ &= \bar{K}(-\xi', -\eta', \tau - \tau' - \tau'') \end{aligned} \tag{A.6c}$$

where for the 1st equality of (A6c) we use Eq. (A.5), i.e.,

$$y_3 - y_2 = (x_3 - \tau'' - \eta') - (\xi' + \tau' + x_2) = \tau - \tau' - \tau'' - \xi' - \eta'$$

and for the 2nd we use (A.3). Thus the B–S equation becomes

$$\bar{D}(\vec{\xi}, \vec{\eta}, \tau) = \bar{D}_0(\vec{\xi}, \vec{\eta}, \tau) + \int d\vec{\xi}'\, d\vec{\eta}'\, d\tau'\, d\tau''\, \bar{D}(\vec{\xi}, \vec{\xi}', \tau')$$

$$\bar{K}(-\vec{\xi}', -\vec{\eta}', \tau - \tau' - \tau'')\, \bar{D}_0(\vec{\eta}', \vec{\eta}, \tau'')$$

Letting $\wedge$ denote the Fourier transform in the τ coordinate only and dropping the bars we have

$$\hat{D}(\vec{\xi}, \vec{\eta}, k) = \hat{D}_0(\vec{\xi}, \vec{\eta}, k) + \int d\vec{\xi}'\, d\vec{\eta}'\, \hat{D}(\vec{\xi}, \vec{\xi}', k)\, \hat{K}(-\vec{\xi}', -\vec{\eta}', k)\, \hat{D}_0(\vec{\eta}', \vec{\eta}, k)$$

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